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Analysis[1]

W. Ted Mahavier $<$ W. S. Mahavier

Lamar University

[1]If each of X and Y is a person, then the relation $X < Y$ indicates that the notes originated with Y and were subsequently modified by X, who takes full responsibility for the current version. While Y is credited with the genesis of the notes, s/he makes no claim to the accuracy of the current version which may or may not reflect her/his original version.

Introduction to Limit Points, Sequences and Cauchy Sequences

These problems are all stated for the vector space of real numbers. The proof techniques and the concepts are valid in any normed vector space, so if you understand the proofs for real numbers, you'll understand the similar statements and proofs in vector spaces.

The word *point* means a real number.

Definition 1. *The statement that the point set O is an* **open interval** *means that there are two points a and b such that O is the set consisting of all points between a and b.*

Definition 2. *The statement that the point set I is a* **closed interval** *means that there are two points a and b such that I is the set consisting of the points a and b and all points between a and b.*

In set notation,

$$(a,b) = \{x : x \text{ is a point and } a < x < b\}$$

and

$$[a,b] = \{x : x \text{ is a point and } a \leq x \leq b\}.$$

We do not use (a,b) or $[a,b]$ in the case $a = b$, although many mathematicians and texts do. We refer to a and b as the **endpoints** of the interval.

Definition 3. *If M is a point set and p is a point, the statement that p is a* **limit point** *of the point set M means that* every *open interval containing p contains a point of M different from p.*

Problem 1. *Show that if M is the open interval (a,b), and p is in M, then p is a limit point of M.*

Problem 2. *Show that if M is the closed interval $[a,b]$, and p is not in M, then p is not a limit point of M.*

Problem 3. *Show that if M is the set of all positive integers, then no point is a limit point of M.*

Problem 4. *Show that if M is the set of all reciprocals of positive integers, then* 0 *(zero) is a limit point of M.*

From here forward, the word *point* may be used to mean either a real number or an ordered pair of real numbers, i.e. a point in the plane.

Definition 4. *The statement that f is a* **function** *means that f is a collection of points in the plane, no two of which have the same first coordinates.*

Definition 5. *If f is a function, then by the* **domain** *of f is meant the point set of all first coordinates of the ordered pairs in f, and by the* **range** *of f is meant the set of all second coordinates of the ordered pairs in f.*

We use the usual notation that if f if a function and x is a number in the domain of f, then $f(x)$ is the number which is the 2^{nd} coordinate of the point of f whose 1^{st} coordinate is x.

Definition 6. *A* ***sequence*** *is a function with domain the natural numbers and with range a subset of real numbers.*

If p is a sequence, then $p = \{(1,p(1)),(2,p(2)),(3,p(3)),\ldots\}$. Since writing p this way is cumbersome and the domain is always the natural numbers, we will denote sequences by listing only the points in the range of the sequence, $p(1),p(2),p(3),\ldots$. We'll further abbreviate this as: $p_1,p_2,p_3,\ldots$. The set $\{p_i : i = 1,2,3,\ldots\}$ denotes the range of the sequence. That is, $\{p_i : i = 1,2,3,\ldots\}$ denotes the point set to which the point x belongs if and only if there is a positive integer n such that $x = p_n$.

Definition 7. *The statement that the point sequence $p_1,p_2,\ldots$* **converges to the point x** *means that if $\varepsilon > 0$ then there is a positive integer N such that if n is a positive integer and $n > N$ then $|p_n - p| < \varepsilon$.*

Definition 8. *The statement that the sequence $p_1,p_2,p_3,\ldots$* **converges** *means that there is a point x such that $p_1,p_2,p_3,\ldots$ converges to x.*

Problem 5. *For each positive integer n, let $p_n = 1 - 1/n$. Prove that the sequence $p_1,p_2,p_3,\ldots$ converges to* 1.

Problem 6. *Show that if the sequence $p_1,p_2,p_3,\ldots$ converges to the point x, and, for each positive integer n, $p_n \neq p_{n+1}$, then x is a limit point of the set which is the range of the sequence.*

Definition 9. *The statement that the sequence $p_1,p_2,p_3,\ldots$ is a* **Cauchy sequence** *means that if ε is a positive number, then there is a positive integer*

N such that if n is a positive integer and m is a positive integer, $n > N$, and $m > N$, then $|p_n - p_m| < \varepsilon$.

Problem 7. *The sequence $p_1, p_2, p_3, \ldots$ is a Cauchy sequence if and only if it is true that for each positive number ε, there is a positive integer N such that if n is a positive integer and $n \geq N$, then $|p_n - p_N| < \varepsilon$.*

Problem 8. *If the sequence $p_1, p_2, p_3, \ldots$ converges to a point x, then $p_1, p_2, p_3, \ldots$ is a Cauchy sequence.*

Problem 9. *If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the set $\{p_1, p_2, p_3, \ldots\}$ is bounded.*

Problem 10. *If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the set $\{p_1, p_2, p_3, \ldots\}$ does not have two limit points.*

Problem 11. *If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the sequence $p_1, p_2, p_3, \ldots$ converges to some point.*

What this shows is that for real numbers, every Cauchy sequence converges. A space is said to be **complete** if every Cauchy sequence converges and so the reals are a complete vector space.